TESTING FOR INDEPENDENCE BETWEEN POISSON PROCESSES

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We observe a stationary multivariate point process, consisting of r parallel univariate point processes, over a time interval, $(0, t)$ and wish to test the hypothesis that its r components are independent and Poisson. Suppose that we have a probability model for the process depending on $r + 1$ unknown parameters, $\mu_1, \ldots \mu_r$ and $\xi \geq 0$ such that if $\xi = 0$ the component processes are independent Poisson processes with rates $\mu_1, \ldots \mu_r$. The required test then becomes a test of the hypothesis $\xi = 0$ against the alternative $\xi > 0$.

In this paper we generalise the results of Davies (1977) by expressing the locally optimal test statistic as the sum of an infinite series, each term being based on a different order of cumulant and, in effect, providing a different piece of information.

It is convenient to consider the multivariate point process as a point process in $R \times \{1, \ldots r\}$ so that a point in this space indicates both the time of occurrence of the point and in which component process it occurs. The outcome of the process may be expressed as an integer valued random measure, N, which assigns a measure 1 to each point of $R \times \{1, \ldots r\}$ at which a *point* of the process occurs. Throughout this paper we suppose that the point process is orderly (only one *point* at a time) and stationary.

Suppose that for each q the q-th order cumulant density can be defined:

$$k(t_1,\ldots t_q\,;\,\xi,\underset{\sim}{\mu})$$
$$= \lim_{\tau\to 0} \tau^{-q} \frac{\partial}{\partial u_1} \cdots \frac{\partial}{\partial u_q} \log E[\prod_1^q u_i^{N(t_i,t_i+\tau)} | \xi,\underset{\sim}{\mu}] \,\Big|_{u_i = 1} \tag{1}$$

for each q distinct points $t_1,\ldots t_q$ in $R \times \{1,\ldots r\}$ where $t_i + \tau$ denotes a point at time τ after a point t_i in the same component process and $E(\cdot\,|\,\xi,\underset{\sim}{\mu})$ denotes expectation under the model with parameters ξ and $\underset{\sim}{\mu} = (\mu_1,\ldots\mu_r)$. The value of k when 2 of its arguments coincide is immaterial and may be defined by continuity. Thus $k(t_1,\ldots t_q\,;0,\underset{\sim}{\mu}) = 0$ if $q > 0$ and μ_j if $q = 1$ and t_1 is in the j-th process. Suppose that

$$k'(t_1,\ldots t_q\,;\underset{\sim}{\mu}) = \frac{\partial}{\partial\xi}k(t_1,\ldots t_q\,;\,\xi,\underset{\sim}{\mu})\,\Big|_{\xi = 0} \tag{2}$$

may be defined. Let $\chi(t_1,\ldots t_q) = 1$ if all of the t_i are distinct and 0 otherwise. Now let $Z^{(t)}$ denote the derivative, with respect to ξ and evaluated at $\xi = 0$, of the log-likelihood of the process observed over $(0,t)$. Thus, if $\underset{\sim}{\mu}$ is known, the locally optimal test of the hypothesis $\xi = 0$ against the alternative $\xi > 0$ rejects the hypothesis for large values of $Z^{(t)}$.

An immediate generalisation of the result of Davies (1977) obtained by differentiating a formula of Kuznetsov and Stratonovich (1956) is that

$$Z^{(t)} = \sum_1^\infty Z_q^{(t)} \tag{3}$$

where

$$Z_q^{(t)} = \frac{1}{q!} \int_{[(0,t)\times\{1,\ldots r\}]^q} \chi(t_1,\ldots t_q)k'(t_1,\ldots t_q;\underset{\sim}{\mu}) \prod_1^q \left[\frac{N(dt_i)}{\mu(t_i)} - dt_i\right] \tag{4}$$

and $\mu(t_i) = \mu_j$ if t_i is in the j-th process. In addition under the hypothesis, that the component processes are independent and Poisson the random variables $t^{-\frac{1}{2}} Z_q^{(t)}$ are asymptotically independently normally distributed with zero mean and variance τ_q^2 given by

$$\tau_q^2 = \frac{1}{q!}\sum_1^r \frac{1}{\mu_j} \int_{[R\times\{1,\dots r\}]^{q-1}} \{k'(0_j, t_2, \dots t_q; \underset{\sim}{\mu})\}^2 \prod_2^q \frac{dt_i}{\mu(t_i)} \tag{5}$$

where 0_j denotes the time 0 in the j-th process. As in Davies (1977), τ_q^2 represents the amount of information in $Z_q^{(t)}$. In the usual situation when $\underset{\sim}{\mu}$ is unknown, the $Z_1^{(t)}$ contains no useful information and must be omitted from the series (3). In the other terms $\underset{\sim}{\mu}$ may, asymptotically, be replaced by an estimate. Also, as in Davies (1977), (4) is asymptotically equivalent to a term based on the q-th order cumulant spectrum.

In practice one would prefer not to have to calculate more than the second order term $Z_2^{(t)}$ in (3), although it is probably feasible to calculate up to the fourth order term. If

$$\tau_2^2 / \sum_2^\infty \tau_q^2 \tag{6}$$

is close to 1 then most of the information is in the second order term and a test based on it would be close to being locally optimal. On the other hand, if (6) is much less than 1 then a test based on $Z_2^{(t)}$ would have low efficiency.

In order to clarify the rather compact notation we look at $Z_2^{(t)}$ in more detail. Suppose $r = 2$; $k_{11}(t)$, $k_{22}(t)$, $k_{12}(t)$ denote the auto- and cross-covariance densities and N_1 and N_2 the random measures corresponding to the 2 series. Then

$$Z_2^{(t)} = \frac{1}{2}\sum_1^2 \frac{1}{\mu_j^2} \int_o^t \int_o^t \chi(t_1, t_2)\, k'_{jj}(t_1 - t_2; \underset{\sim}{\mu}) \prod_1^2 \{N_j(dt_i) - \mu_j dt_i\}$$

$$+ \frac{1}{\mu_1\mu_2} \int_o^t \int_o^t k'_{12}(t_1 - t_2; \underset{\sim}{\mu})\{N_1(dt_1) - \mu_1 dt_1\}\{N_2(dt_2) - \mu_2 dt_2\} \tag{7}$$

and

$$\tau_2^2 = \frac{1}{2}\sum_1^2 \frac{1}{\mu_j^2} \int_o^t \{k'_{jj}(s; \underset{\sim}{\mu})\}^2 ds + \frac{1}{\mu_1\mu_2} \int_o^t \{k'_{12}(s; \underset{\sim}{\mu})\}^2 ds.$$

<u>Example</u>

Consider 2 processes; the first one being Poisson with rate μ_1 and the second being the superposition of a Poisson process with rate μ_2 and a process generated as follows: with probability ξ each *point*, T, of the first process independently generates a *point* at a time uniformly distributed in $(T, T + \Delta)$ where Δ is a known constant. Then the processes are separately Poisson so $k_{11} = k_{22} = 0$. However, $k_{12}(t;\xi,\underset{\sim}{\mu}) = \xi\mu_1/\Delta$ if $0 < t < \Delta$ and 0 otherwise. The higher order terms are zero and so the test based on just the second term of (7) is locally optimal.

BIBLIOGRAPHY

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Key words:

Cumulant density, Independence, Locally optimal test, Poisson process.

SUMMARY

The results of Davies (1977), generalised to a multivariate point process, provide a test of the independence of Poisson processes.

RESUMÉ

Les résultats de Davies (1977), généralisés à un processus p nctuel à plusieurs variables, fournissent un test de l'indépendance des processus de Poisson.